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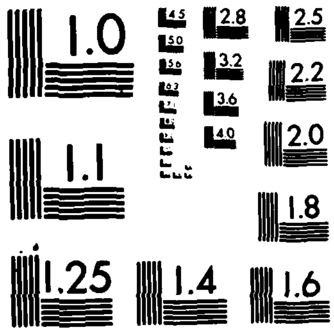

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COMMENTS ON A PROBLEM OF CHERNOFF AND PETKAU

by

Michael L. Hogan
Columbia University

TECHNICAL REPORT NO. 27
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Abstract

A new method is used to study the optimal stopping set corrected for discreteness introduced by Chernoff and studied by Chernoff and Petkau. The discrete boundary is asymptotically the optimal boundary for a Wiener process translated downward by a constant amount. This amount is shown to be an "excess over the boundary" term, and this method yields it as a simple integral involving the characateristic function of the random walk.

Key Words: Excess Over the Boundary, Optimal Stopping, Wiener Process, Corrected Diffusion Approximations.

This paper consists of an application of ideas about boundary crossing by random walks to problems considered by Chernoff [1965], and Chernoff and Petkau [1976]. In [2], a Bayes test for the sign of a normal mean leads, in the diffusion limit, to an optimal stopping problem for the Wiener process, whose solution can be shown to be given by stopping the first time a Wiener process crosses a certain boundary. The boundary is given as the solution to a free boundary problem. In [1], the discrete version of the same problem is considered. Suppose stopping is permitted only at times $n\delta$, $n = 0, 1, \ldots$. Once again it is possible to show that optimal policy is to stop when the Wiener process crosses a certain boundary at the permitted times. The question is: what is the relation between the boundary $\tilde{x}(t)$ of the unrestricted problem, and $\tilde{x}_\delta(t)$ of the restricted problem? Chernoff showed that $\tilde{x}_\delta(t) = \tilde{x}(t) + \hat{Z}\ \delta^{1/2} + o(\delta^{1/2})$, where, according to [1], $\hat{Z} \doteq -.582$.

The key step in the proof of this is to introduce an auxiliary problem in which a Wiener process is started at a point (z, t), $t < 0$, each observation costs a dollar, no payoff is made if stopping occurs before time 0, and at $t = 0$ a payoff $Z^2\ 1_{(Z<0)}$ is received, and stopping is permitted only at times $t = 0, -1, \ldots$. For this problem the optimal stopping boundary can be shown to be increasing and contained in $[-1, 0]$, and therefore it has a limit as $t \to \infty$ which turns out to be $\hat{Z}$.

In Chernoff and Petkau [1976] the solution to the auxiliary problem is considered for a family of dichotomous random variables depending on a parameter p, and continuity of $\hat{Z}$ as a function of p is established, as well as a method for calculating $\hat{Z}$ when p is rational.

This comment is a result of a question of Professor David Siegmund after Professor Chernoff's talk at the Kiefer-Neyman conference at Berkeley, June 1983. Professor Siegmund remarked that $\hat{Z} = -ER$, where R is a random

variable whose distribution is the same as that of the asymptotic excess over the boundary for the normal random walk (see below for precise definition). Chernoff's analytic machinery, in particular the probabilistic interpretation of the solution of a Wiener-Hopf equation (see Spitzer [1957, 1960]) shows this to be the case. The first part of this paper presents a more direct connection between the role of $\hat{Z}$ as the expected asymptotic excess over the boundary, and $\hat{Z}$ as the solution to the auxiliary problem. The second part identifies the quantity $\hat{Z}$ in the auxiliary problem for arbitrary random walk with $E|S_1|^3 < \infty$, and establishes the continuity as a function of p in Chernoff and Petkau's family of dichotomous random variables A benefit of this method is that $\hat{Z}$ can be computed as a one dimensional integral involving the characteristic function of S_1.

Here is some notation that will be used in the proof of Theorem 1; however be warned that due to the extremely heavy notational demands imposed by Theorem 2 these definitions will only hold for the proof of Theorem 1. Let S_i be a random walk with $ES_1 = 0$, $ES_1^2 = 1$, $E|S_1|^3 < \infty$. Let $\tau_a = \inf\{n : S_n > a\}$, $R_a = S_{\tau_a} - a$, $\tau_+ = \tau_0$, and R be a random walk such that $\mathcal{L}(R) = \lim_{a\to\infty} \mathcal{L}(R_a)$, where $\mathcal{L}(X)$ denotes the distribution of the random variable X, and where $a \to \infty$ through numbers of the form nd if S_1 is arithmetic with span d. This limit is known to exist from renewal theory and it is also known that $\lim_{a\to\infty} ER_1 = ER$. It is easy to show that for nonarithmetic S_1, $ER = \frac{ES_{\tau_+}^2}{2ES_{\tau_+}}$, while for arithmetic S_1 with span d, $ER = \frac{ES_{\tau_+}^2}{2ES_{\tau_+}} - \frac{d}{2}$, as $a \to \infty$ through multiples of d. The limit as $t \to \infty$ of the optimal boundary of the auxiliary problem will be denoted $\hat{Z}$. (An *arithmetic* random variable takes values in a discrete subgroup of $\Re$.) A *lattice* is a coset of a discrete subgroup of $\Re$.

The auxiliary problem is as defined above, except that the random walk generated by S_1 is run rather than a Wiener process.

Theorem 1. $\hat{Z} = -\frac{ES_{\tau_+}^2}{2ES_{\tau_+}}$.

Remark. According to Siegmund [4], Chapter X

$$\hat{Z} = \frac{1}{6}ES_1^3 - \frac{1}{\pi}\int_0^\infty t^{-2} Re \log\{2[1-f(t)]/t^2\}dt,$$

where Re denotes real part, and $f(t) = Ee^{itS_1}$.

Proof. Suppose the optimal boundary is given by $Z(y)$, and that the random walk starts at time $-n_0$. Let $T = \inf\{n : S_n > Z(n-n_0)\}$. Recall that the same boundary is optimal regardless of starting position. Then, Chernoff and Petkau show that the problem is equivalent to picking $Z(y)$ to minimize

$$E(S_T^2;\ T<0) + E(S_T^2;\ T=0, S_T<0). \tag{1}$$

Now, assume that S_1 is nonlattice. A starting position can be chosen so that S_n crosses $Z(z)$ when $|Z - \hat{Z}| < \epsilon$, with probability $> 1-\epsilon$. It is easy to establish uniform integrability of S_T^2 independent of starting position. Moreover, a starting position can be picked so that $\hat{Z}$ is very much larger than the starting position. The small variation of Z makes it clear that S_T has approximately the distribution $\hat{Z} + R$. This can be made rigorous using simplified versions of an argument presented in Hogan [1984]. Lemma 4.1 stochastically bounds S_T^2, and the argument of Theorem 4.1 of that work establishes the statement about the distribution of S_T. The idea in this case is clear: the nearly constant boundary Z can be replaced by the constant boundary $\hat{Z}$. Therefore, by (1), the problem is equivalent to minimizing

$$E(\hat{Z}+R)^2 + \eta$$

where η can be made arbitrarily small by a suitable choice of starting position. The solution to this problem is to take $\hat{Z} = -ER$.

In the arithmetic case, let d be the span of the distribution of S_1. A starting position determines a lattice on which the random walk lives. Let $\Delta(\hat{Z})$ denote the distance to the closest lattice value smaller than $\hat{Z}$. Excess over $\hat{Z} - \Delta(\hat{Z})$ has a known asymptotic distribution. The problem is essentially to minimize $E(\hat{Z}+Y)^2$, where again Y is the excess over the optimal boundary. Now, though, $Y + \Delta(\hat{Z})$ has the fixed, known limiting distribution $\mathcal{L}(R)$. Consequently

$$\begin{aligned} E(\hat{Z}+Y)^2 &= E(Y+\hat{Z}+\Delta(\hat{Z})-\Delta(\hat{Z}))^2 \\ &= E(Y+\Delta(\hat{Z})-ER+ER+\hat{Z}-\Delta(\hat{Z})) \\ &= E(Y+\Delta(\hat{Z})-ER)^2+(ER+\hat{Z}-\Delta(\hat{Z}))^2. \end{aligned}$$

The problem of minimizing this is equivalent to that of minimizing $(ER + \hat{Z} - \Delta(\hat{Z}))^2$, as the first term is approximately constant, independent of $\hat{Z}$. It is easy to see that this is done by taking $\hat{Z} = -ER + \frac{d}{2}$. The proof is finished by observing that, by the arithmetic renewal theorem, $ER - \frac{d}{2} = \frac{ES^2_{\tau_+}}{2ES_{\tau_+}}$.

Chernoff and Petkau consider the same stoppping problem for a family of random walks generated by dichotomous random variables

$$S_1^p = \begin{cases} (p/1-p)^{1/2} = b(p) & \text{with probability } p \\ -((1-p)/p)^{1/2} = -a(p) & \text{otherwise} \end{cases}$$

$ES_1^p = 0$, $E(S_1^p)^2 = 1$. It is possible to give a pathwise construction of this process that shows that up to any stochastically bounded time t the process is continuous in probability as a function of p, i.e., for a fixed p_0, and for p close to p_0 the paths up to time t can be made arbitrarily close to those of p_0 except for a set of small probability. This seems reasonably clear so proof is omitted. Another fact about these processes that should be noted is that S_1^p is nonarithmetic iff p

is irrational. Here is some notation that will be necessary:

$$
\begin{aligned}
\tau_+^p &= \inf\{n : S_n^p > 0\} \text{ and by induction,} \\
\tau_+^{p(1)} &= \tau_+^p \\
\tau_+^{p(n)} &= \inf\{j > \tau_+^{p(n-1)} : S_j^p > S_{\tau_+^{p(n-1)}}\} \text{ for } n > 1, \\
\bar{\tau}_+^p &= \inf\{n : S_n^p \geq 0\}, \\
\tau_\epsilon^p &= \inf\{n : S_n^p > \epsilon\}.
\end{aligned}
$$

If p is irrational, $\bar{\tau}_+^p = \tau_+^p$ with probability 1, for $P\{S_n^p = 0\} = 0 \, \forall n$.

To simplify the notation quantities like $E(S_{\tau_+^{p_0}}^{p_0})$ will be denoted $E_{p_0}(S_{\tau_+})$ when this can be done unambiguously.

Theorem 2.

$$\frac{E_p S_{\tau_+}^2}{E_p S_{\tau_+}}$$

is a continuous function of p.

Proof. Note first that, by the path continuity property, continuity is obvious at a value of p for which $P\{\bar{\tau}_+^p = \tau_+^p\}$, i.e. p is irrational. In this case the numerator and denominator are separately continuous. This accomplishes the main purpose of Chernoff and Petkau's result, which is to show that a calculation which can be done for p rational can be extended to irrational p by continuity. Therefore, it suffices to establish continuity at some fixed, rational p.

The heuristic for why this works is that no amount of messing around near zero and continuing by the random walk can affect the ratio considered here, although numerator and denominator seperately can change. The simplest example of this phenomenon is

$$\frac{E_{p_0} S_{\bar{\tau}_+}^2}{E_{p_0} S_{\bar{\tau}_+}} = \frac{E_{p_0} S_{\tau_+}^2}{E_{p_0} S_{\tau_+}}.$$

This will be used below. It is an example of the heuristic because to get from $\bar{\tau}_+^{p_0}$ to $\tau_+^{p_0}$ the random walk hits *at* 0 and then goes on to perform an identical copy of $\tau_+^{p_0}$. Formally

$$\begin{aligned} E_{p_0}(S_{\tau_+}) &= E_{p_0}(S_{\bar{\tau}_+};\ \bar{\tau}_+ = \tau_+) \\ &= E_{p_0}(S_{\bar{\tau}_+}) + P_{p_0}\{\bar{\tau}_+ < \tau_+\} E_{p_0}(S_{\tau_+}), \end{aligned}$$

or

$$E_{p_0}(S_{\tau_+}) = \frac{E_{p_0}(S_{\bar{\tau}_+})}{P_{p_0}\{\bar{\tau}_+ = \tau_+\}}.$$

Similarly,

$$E_{p_0}(S_{\tau_+}^2) = \frac{E_{p_0}(S_{\bar{\tau}_+}^2)}{P_{p_0}\{\bar{\tau}_+ = \tau_+\}},$$

which establishes the claim.

The second example of this heuristic is

$$E_p(S_{\tau_\epsilon}) = c(\epsilon)\, E_p\, S_{\tau_+} + 0(\epsilon)$$

and

$$E_p(S_{\tau_\epsilon}^2) = c(\epsilon)\, E_p\, S_{\tau+}^2 + 0(\epsilon).$$

To see this write

$$\begin{aligned} E_p(S_{\tau_\epsilon}) &= E(S_{\tau_+};\ \tau_+ = \tau_\epsilon) + \int_0^\epsilon E(S_{\tau_{\epsilon-x}})\, P\{S_{\tau_+} \in dx\} \\ &= E(S_{\tau_+}) + 0(\epsilon) + \int_0^\epsilon E(S_{\tau_+};\ \tau_{\epsilon-x} = \tau_+)\, P\{S_{\tau_+} \in dx\} \\ &\quad + \int_0^\epsilon \int_0^{\epsilon-x} E(S_{\tau_{\epsilon-x-y}})\, P\{S_{\tau_+} \in dy\} P\{S_{\tau_+} \in dx\} \\ &= E(S_{\tau_+}) + 0_1(\epsilon) + (E(S_{\tau_+}) + 0_2(\epsilon))\, P\{\tau_\epsilon > \tau_+\} \\ &\quad + \int_0^\epsilon \int_0^{\epsilon-z} E(S_{\tau_{\epsilon-x-z}})\, P\{S_{\tau_+} \in dz\}\ p\{S_{\tau_+} \in dz\} \end{aligned}$$

where $|0_1(\epsilon)|$, $|0_2(\epsilon)| < \epsilon$. Continuing in this manner establishes the claim with $c(\epsilon) = \sum_{n=1}^{\infty} P\{\tau_\epsilon > \tau_+^{(n)}\}$. $ES^2_{\tau_\epsilon}$ follows analogously.

Now the result follows easily. The path continuity property makes it clear that

$$(S^q_{\tau^q_+} 1_{(\tau^{p_0}_+ = \tau^{p_0}_+)} \to S^{p_0}_{\tau^{p_0}_+} 1_{(\tau^{p_0}_+ = \tau^{p_0}_+)}).$$

Convergence takes place as long as $S^{p_0}_{\tau^{p_0}_+} > 0$. For the other paths, $S^q_{\tau^{p_0}_+}$ is close to zero, possibly above and possibly below. By these two observations

$$E\, S^q_{\tau^q_+} \doteq E\, S^{p_0}_{\tau^{p_0}_+} + E\, S^q_{\tau^q_+} \int_{-\delta}^{0} c(\epsilon)\, p\{S^q_{\tau^{p_0}_+} \in dx\} + o(1),$$

where $o(1) \to 0$ as $q \to p$, and $E(S^q_{\tau^q_+})^2$ satisfies a similar equation. The fact about the ratios is now obvious.

Here are some consequences of Theorem 1.

(1) (Part of [2], Theorem 3.3). $\hat{Z} \geq -\frac{b}{2}$.

Proof. Since $S_{\tau_+} \leq b$,

$$\hat{Z} = -\frac{ES^2_{\tau_+}}{2\, ES_{\tau_+}} \geq -\frac{b\, ES_{\tau_+}}{2\, ES_{\tau_+}} = -\frac{b}{2},$$

where $b = b(p)$ as above. Jensen's inequality produces the upper bound

$$\hat{Z} \leq -\frac{(ES_{\tau_+})^2}{2\, ES_{\tau_+}} \leq -\frac{b}{2} \cdot (1-p).$$

(2) If $p = \frac{1}{n}$, $\hat{Z} = -\frac{b}{2}$ ([2], Section 4, p. 882). For then $-a > b$ and $\frac{b}{a}$ is an integer. Under these conditions it is easy to see that $S_{\tau_+} = b$. For $p = \frac{n-1}{n}$, n an integer, $\hat{Z} = -\frac{b}{2}$ also holds for a symmetric reason.

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